

## Diverging probability-density functions for flat-top solitary waves

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We investigate the statistics of flat-top solitary wave parameters in the presence of weak multiplicative dissipative disorder. We consider first propagation of solitary waves of the cubic-quintic nonlinear Schrödinger equation (CQNLSE) in the presence of disorder in the cubic nonlinear gain. We show by a perturbative analytic calculation and by Monte Carlo simulations that the probability-density function (PDF) of the amplitude  $\eta$  exhibits loglognormal divergence near the maximum possible amplitude  $\eta_m$ , a behavior that is similar to the one observed earlier for disorder in the linear gain [A. Peleg *et al.*, Phys. Rev. E **72**, 027203 (2005)]. We relate the loglognormal divergence of the amplitude PDF to the superexponential approach of  $\eta$  to  $\eta_m$  in the corresponding deterministic model with linear/nonlinear gain. Furthermore, for solitary waves of the derivative CQNLSE with weak disorder in the linear gain both the amplitude and the group velocity  $\beta$  become random. We therefore study analytically and by Monte Carlo simulations the PDF of the parameter  $p$ , where  $p = \eta / (1 - \varepsilon_s \beta / 2)$  and  $\varepsilon_s$  is the self-steepening coefficient. Our analytic calculations and numerical simulations show that the PDF of  $p$  is loglognormally divergent near the maximum  $p$  value.

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### I. INTRODUCTION

Flat-top solitary waves are coherent patterns, which exist as a result of a balance between dispersion/diffraction and competing nonlinearities, where the low order nonlinearity is “focusing” while the high order nonlinearity is “defocusing” [1–3]. When the intensity of the field is relatively small, the low order nonlinearity is dominant, and consequently, the solitary waves are narrow and have a shape that is similar to that of conventional solitons. However, when the intensity increases, the high order nonlinearity becomes dominant and leads to the broadening of the pulse shape and to the generation of a typical table-top pattern. Flat-top solitary waves appear as solutions to nonlinear wave models in many areas of physics, including nonlinear optics [1,4,5] fluid dynamics [3,6], and plasma physics [7,8]. As a result, they have been the subject of intensive research efforts in recent years. The interest in flat-top solitary waves is further enhanced since they are used in pattern formation theory to explain the emergence of fronts (kinks) from localized coherent structures such as solitons and solitary waves [9,10]. Many of the systems in which flat-top solitary waves appear can be influenced by processes involving noise or disorder. When the disorder is strong the solitary waves are usually destroyed, whereas, when it is weak, the solitary waves can form and evolve. In the latter case one is mainly concerned with the statistics of the solitary wave parameters.

In this study we focus attention on an important type of disorder, which we call multiplicative dissipative disorder. This type of disorder is characterized by the following properties: (1) the disorder affects the amplitude of the solitary wave in first order of the perturbation; (2) the disorder term in the nonlinear wave model is multiplicative. Dissipative disorder can appear in systems described by nonlinear wave equations in various forms. Two of the most common forms are disorder in the linear gain/loss coefficient and disorder in

the cubic nonlinear gain/loss coefficient. Disorder in the linear gain coefficient can appear in optical fiber communication systems due to randomness in the gain of amplifiers that are positioned along the fiber line to compensate for the loss [11]. Moreover, such disorder appears in massive multichannel optical fiber transmission as a result of the interplay between Raman-induced energy exchange in pulse collisions and bit pattern randomness [12–16]. In this case, the disorder can lead to relatively high bit-error-rate values and intermittent dynamics of pulse parameters [16–18]. We point out that in all the studies reported in Refs. [11–18] weak disorder was considered. In addition, both linear and cubic nonlinear disorder in the gain can emerge in an active nonlinear medium due to random variations with distance in the linear/nonlinear gain/loss coefficient.

We consider two nonlinear wave models, which possess flat-top solitary wave solutions: the cubic-quintic nonlinear Schrödinger equation (CQNLSE) and the derivative CQNLSE (DCQNLSE). A third model, the extended Korteweg-de Vries (eKdV) equation, is also briefly discussed. The CQNLSE is a simple nonintegrable extension of the cubic nonlinear Schrödinger equation (CNLSE) possessing solitary wave solutions. The CQNLSE describes a variety of physical systems including pulse propagation in semiconductor-doped optical fibers [1,4,19–23], laser-plasma interaction [7,24], and Bose-Einstein condensates [25–28]. Another important reason for the interest in the CQNLSE is that due to its nonintegrability it allows one to observe dynamical effects that do not exist in the CNLSE, e.g., emission of continuous radiation in two-soliton collisions [29,30]. Furthermore, the cubic-quintic complex Ginzburg-Landau equation, which is a generalization of the CQNLSE including dissipative terms, is known to describe even a wider range of physical systems, including, for example, convection and pattern formation in fluids [9,10,31–34] and mode-locked lasers [5,35–38].

The derivative CNLSE is an extension of the CNLSE that, in the context of nonlinear optics, takes into account the effects of self-steepening [19,39–41]. For short optical pulses propagating in semiconductor-doped fibers both quintic nonlinearity and self-steepening are important. Consequently, one expects that the derivative CQNLSE, which takes into account both effects, would provide a more accurate description of the propagation in this case. We note that a variant of the derivative CNLSE is known to describe propagation of Alfvén waves in magnetized plasmas [42–44]. Moreover, it was recently shown theoretically and experimentally that a variant of the DCQNLSE accurately describes propagation of high-intensity pulses in cascaded-quadratic nonlinear media [45].

The eKdV equation, which is also known as the Gardner equation, is an integrable model that describes interfacial waves in a two-layer system [46–48] as well as stratified shear flow in the ocean [3,49,50]. It provides a possible explanation to observations of large-amplitude flat-top solitary waves in coastal zones [51–53].

In a previous work [54] we studied the effects of weak disorder in the linear gain coefficient on solitary waves of the CQNLSE. We showed analytically (by employing an adiabatic perturbation method) and by Monte Carlo simulations that the probability-density function (PDF) of the solitary wave amplitude has a loglognormal diverging form in the vicinity of the maximum possible amplitude. Since solitary waves with amplitude values close to the maximum possible amplitude have a table-top shape, this finding means that the amplitude PDF of flat-top solitary waves exhibits loglognormal divergence. We also conjectured that similar loglognormal divergence should be observed for disorder in the nonlinear gain. However, the full analytic form of the amplitude PDF for disorder in the nonlinear gain was not obtained and the conjecture was not tested by numerical simulations. Thus, the important question concerning the generality of the loglognormal divergence of the amplitude PDF for flat-top solitary waves was not fully answered, even for the CQNLSE. Furthermore, it is not clear whether loglognormal divergence can be observed in other nonlinear wave models possessing solitary wave solutions. In the current paper we address these questions in detail. We start by considering propagation of solitary waves of the CQNLSE in the presence of weak disorder in the cubic nonlinear gain. The case of disorder in the cubic nonlinear gain is particularly important since cubic gain/loss is very common in many systems described by the complex Ginzburg-Landau equation (see, for example, Ref. [33] and references therein). We calculate the amplitude PDF analytically by employing an adiabatic perturbation method and validate its loglognormal divergence by Monte Carlo simulations. We then turn to study propagation of solitary waves of the DCQNLSE in the presence of weak disorder in the linear gain. In this case both the amplitude and the group velocity randomly vary during propagation. We therefore study the PDF of a new parameter, which is the ratio between the amplitude and a linear function of the group velocity. We find that the PDF of this new parameter is loglognormally divergent near the parameter's maximum value. We conclude by a brief discussion of the dynamic mechanism responsible for the loglognormal diver-

gence of the PDFs, and of the possibility to observe similar statistical behavior in the eKdV model.

The material in the rest of the paper is organized as follows. In Sec. II, we study propagation of solitary waves of the CQNLSE in the presence of disorder in the cubic nonlinear gain. In Sec. III, we investigate propagation of solitary waves of the DCQNLSE in the presence of disorder in the linear gain. Sections IV and V are reserved for discussion and conclusions, respectively. In Appendix A, we describe a method for identifying loglognormal divergence in numerical data. Finally, Appendixes B and C are devoted to calculation of the PDF by employing the Fokker-Planck approach and by working within Itô's interpretation.

## II. CUBIC-QUINTIC NLSE WITH DISORDER IN THE CUBIC GAIN/LOSS COEFFICIENT

Consider the dynamics described by the CQNLSE with disorder in the cubic gain/loss coefficient,

$$i\partial_z\psi + \partial_t^2\psi + 2|\psi|^2\psi - \varepsilon_q|\psi|^4\psi = i\varepsilon\xi(z)|\psi|^2\psi, \quad (1)$$

where the disorder  $\xi(z)$  is zero in average and short correlated in  $z$ ,

$$\langle \xi(z) \rangle = 0, \quad \langle \xi(z)\xi(z') \rangle = D\delta(z-z'). \quad (2)$$

In the context of nonlinear optical waveguides  $\psi$  is proportional to the envelope of the electric field,  $z$  is the propagation distance,  $t$  is a retarded time,  $\varepsilon_q$  is the quintic nonlinearity coefficient,  $0 < \varepsilon \ll 1$  is the cubic gain coefficient and  $D$  is the disorder intensity. The terms  $\varepsilon_q|\psi|^4\psi$  and  $i\varepsilon\xi(z)|\psi|^2\psi$  describe the effects of quintic nonlinearity and disorder in the cubic nonlinear gain/loss coefficient, respectively. When  $\varepsilon = 0$ , Eq. (1) supports stable solitary wave solutions of the form [55]  $\psi_s(t, z) = \Psi_s(x)\exp(i\chi)$ , where

$$\Psi_s(x) = \frac{\sqrt{2}\eta}{[(1 - \eta^2/\eta_m^2)^{1/2}\cosh(2x) + 1]^{1/2}}, \quad (3)$$

$\eta_m \equiv (4\varepsilon_q/3)^{-1/2}$ ,  $\chi = \alpha + \beta(t-y) + (\eta^2 - \beta^2)z$ , and  $x = \eta(t-y - 2\beta z)$ . In these relations the parameters  $\eta$ ,  $\beta$ ,  $\alpha$ ,  $y$  are related to the amplitude, frequency, phase, and position of the solitary wave, respectively. Note that the solitary wave solution  $\psi_s$  exists provided that  $\eta < \eta_m$ .

We study the dynamics of the solitary wave  $\psi_s$  as described by Eq. (1). Since we are interested in flat-top solitary waves we focus attention on the case  $\varepsilon_q > 0$ . We also assume that  $4D\varepsilon^2z \ll 1$ , so that for most of the disorder realizations the dynamics of the solitary wave amplitude is not yet influenced by the  $O(\varepsilon^2)$  radiation instability effects [9,56]. The dynamics of the parameter  $\eta$  is obtained by using energy balance considerations,

$$\partial_z \int_{-\infty}^{\infty} dt |\psi|^2 = 2\varepsilon\xi(z) \int_{-\infty}^{\infty} dt |\psi|^4. \quad (4)$$

In order to solve Eq. (4) we employ the adiabatic perturbation method, which has been extensively used in previous studies of the CQNLSE, see, for example, Ref. [9]. This calculation yields

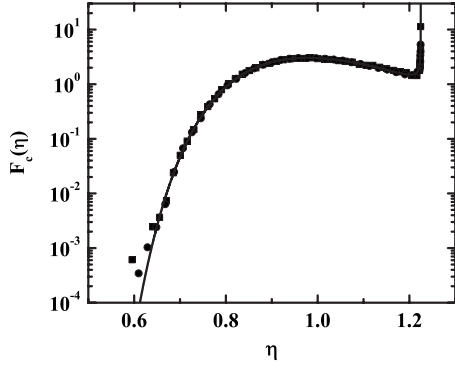


FIG. 1. The probability-density function  $F_c(\eta)$  at  $z=10$  for  $D=3$ ,  $\varepsilon_q=0.5$ ,  $\varepsilon=0.03$  and  $\eta(0)=1$ . The solid curve corresponds to the analytic result obtained by using Eqs. (8) and (10). The squares represent the result of Monte Carlo simulations with Eq. (1), while the circles stand for the result of Monte Carlo simulations with Eq. (8).

$$\frac{d}{dz} \left[ \operatorname{arctanh} \left( \frac{\eta}{\eta_m} \right) \right] = 4\varepsilon\eta_m^2\xi(z) \left[ \operatorname{arctanh} \left( \frac{\eta}{\eta_m} \right) - \frac{\eta}{\eta_m} \right]. \quad (5)$$

Furthermore, within the framework of Stratonovich's interpretation [57–59] Eq. (5) can be transformed into

$$\frac{d\eta}{dz} = 4\varepsilon\xi(z)(\eta_m^2 - \eta^2) \left[ \eta_m \operatorname{arctanh} \left( \frac{\eta}{\eta_m} \right) - \eta \right]. \quad (6)$$

Changing variables from  $\eta$  to  $v$ , where

$$v = \int \frac{d\eta}{(\eta_m^2 - \eta^2)[\eta_m \operatorname{arctanh}(\eta/\eta_m) - \eta]}, \quad (7)$$

we obtain the equation  $dv/dz=4\varepsilon\xi(z)$ , whose solution is

$$v(z) - v(0) = \int_{\eta(0)}^{\eta(z)} \frac{d\eta}{(\eta_m^2 - \eta^2)[\eta_m \operatorname{arctanh}(\eta/\eta_m) - \eta]} = 4\varepsilon X(z), \quad (8)$$

where  $X(z) = \int_0^z dz' \xi(z')$ . Notice that  $X(z)$  can be regarded as a sum over many independent random variables. Consequently, according to the central limit theorem its PDF approaches a Gaussian PDF of the form

$$\tilde{F}(X) = (2\pi Dz)^{-1/2} \exp[-X^2/(2Dz)]. \quad (9)$$

Equation (8) defines a monotonously increasing function  $X=q(\eta)$  on  $0 \leq \eta < \eta_m$ . Changing variable from  $X$  to  $\eta$  while employing Eqs. (8) and (9) we obtain that the PDF of  $\eta$  is given by

$$F_c(\eta) = \frac{(32\pi D\varepsilon^2 z)^{-1/2} \exp[-q^2(\eta)/(2Dz)]}{(\eta_m^2 - \eta^2)[\eta_m \operatorname{arctanh}(\eta/\eta_m) - \eta]} \quad (10)$$

for  $0 < \eta < \eta_m$  and  $F_c(\eta)=0$  elsewhere. To calculate the value of  $F_c(\eta)$  for a given  $\eta$  one numerically solves Eq. (8) for  $X=q(\eta)$  and then substitutes the result into Eq. (10). The graph of  $F_c(\eta)$  obtained by this calculation is shown in Figs. 1 and 2.

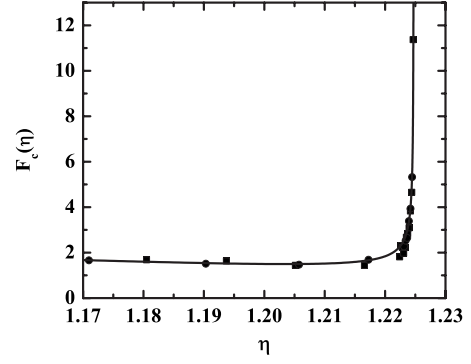


FIG. 2. Blowup of the data in Fig. 1 in the vicinity of  $\eta_m$  showing the divergence of  $F_c(\eta)$ .

As can be seen from Figs. 1 and 2,  $F_c(\eta)$  diverges in the vicinity of  $\eta_m$ . In order to characterize the divergence we obtain an approximate analytic solution of Eq. (8) for  $\eta$  near  $\eta_m$ . We first write the integral in Eq. (8) as a sum of two integrals,

$$4\varepsilon X(z) = \int_{\eta(0)}^{\eta(\bar{z})} \frac{d\eta}{(\eta_m^2 - \eta^2)[\eta_m \operatorname{arctanh}(\eta/\eta_m) - \eta]} + \int_{\eta(\bar{z})}^{\eta(z)} \frac{d\eta}{(\eta_m^2 - \eta^2)[\eta_m \operatorname{arctanh}(\eta/\eta_m) - \eta]}, \quad (11)$$

where  $\eta(\bar{z})$  is a constant satisfying  $\eta(\bar{z}) < \eta(z) < \eta_m$ , and both  $\eta(\bar{z})$  and  $\eta(z)$  are close to  $\eta_m$ . Since both limits of the first integral on the right hand side of Eq. (11) are constants this integral is a constant that we denote by  $c_1$ . We also denote  $\delta\eta = \eta_m - \eta$  and notice that  $0 < \delta\eta(z)/\eta_m < \delta\eta(\bar{z})/\eta_m \ll 1$ . We can therefore expand the integrand in the second integral on the right hand side of Eq. (11) about  $\delta\eta=0$ , keeping terms up to order  $\delta\eta$  in the denominator. This calculation yields

$$4\varepsilon X(z) \simeq c_1 + \frac{1}{\eta_m^2} \int_{\delta\eta(\bar{z})}^{\delta\eta(z)} \frac{d\delta\eta}{\delta\eta \ln \left( \frac{e^2 \delta\eta}{2\eta_m} \right)}. \quad (12)$$

Integrating over  $\delta\eta$  we arrive at

$$X(z) \simeq \ln \left\{ -\ln \left[ \frac{e^2 \delta\eta(z)}{2\eta_m} \right] \right\} / \tilde{c} / (4\varepsilon\eta_m^2), \quad (13)$$

where  $\tilde{c}$  is another constant. Since the normally distributed random variable  $X(z)$  is related to  $\delta\eta(z)$  via a double logarithm, we say that  $\delta\eta(z)$  is loglognormally distributed. Using Eqs. (9) and (13), and changing variables from  $X$  to  $\delta\eta$  we obtain

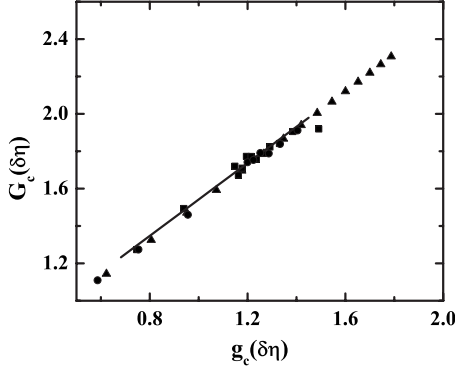


FIG. 3.  $G_c(\delta\eta)$  vs  $g_c(\delta\eta)$  for the same parameters considered in Figs. 1 and 2. The triangles represent the analytic prediction of Eqs. (8) and (10), the squares stand for the result of numerical simulations with Eq. (1) and the circles correspond to the result obtained by Monte Carlo simulations with Eq. (8). The solid line is a linear fit of the squares with a slope of 0.97.

$$F_c(\eta)|_{\eta \approx \eta_m} \approx \left\{ (32\pi D \epsilon^2 z)^{1/2} \eta_m^2 \delta\eta \left| \ln \left[ \frac{e^2 \delta\eta}{2\eta_m} \right] \right| \right\}^{-1} \times \exp \left\{ - \frac{\ln^2[-\ln[(e^2 \delta\eta)/(2\eta_m)]/\bar{c}]}{32D \epsilon^2 \eta_m^4 z} \right\}, \quad (14)$$

from which it follows that the divergence of  $F_c(\eta)$  near  $\eta_m$  is loglognormal.

To check the analytic predictions given by Eqs. (10) and (14) we performed Monte Carlo simulations with Eq. (1) with  $1.09 \times 10^5$  disorder realizations. We considered the parameter values  $D=3$ ,  $\epsilon_q=0.5$  (corresponding to  $\eta_m \approx 1.22474$ ) and  $\epsilon=0.03$ . The initial condition was taken in the form of the solitary wave  $\psi_s$  with  $\eta(0)=1$ ,  $\beta(0)=0$ ,  $y(0)=0$ , and  $\alpha(0)=0$ . We carried out the simulations up to a final distance  $z_f=10$ , for which the disorder strength is  $4D\epsilon^2 z_f=0.3$ . Equation (1) was integrated by employing a split-step method that is of fourth order with respect to the  $z$ -step  $dz$  [60]. The linear part  $i\partial_z \psi = -\partial_t^2 \psi$  was advanced efficiently via an evaluation of the operator exponential in Fourier space and the nonlinear part  $i\partial_z \psi = \epsilon_q |\psi|^4 \psi + (i\epsilon \xi(z) - 2)|\psi|^2 \psi$  was advanced via a fourth-order Runge-Kutta scheme. To overcome numerical errors resulting from radiation emission and the use of periodic boundary conditions we applied the method of artificial damping in the vicinity of the boundaries of the computational domain. (See Refs. [30,54,61] for other examples where the same method was successfully used). The size of the domain was taken to be  $-L \leq t \leq L$  with  $L=16\pi$  so that the absorbing layers do not affect the dynamics of the solitary waves. The  $t$  step and  $z$  step were taken as  $\Delta t=0.01$  and  $\Delta z=0.001$ , respectively.

Figure 1 shows the  $\eta$ -PDF obtained by the simulations as well as the analytic prediction obtained with Eqs. (8) and (10) and the PDF obtained by Monte Carlo simulations with Eq. (8). Figure 2 shows a blowup of the same data in the neighborhood of  $\eta_m$ . The good agreement between the three results strongly indicates that the divergence is indeed loglognormal. In Fig. 3 we present a more sensitive analysis of

this divergence that is based on the procedure described in Appendix A for detecting loglognormal divergence in numerical data. Following this procedure we plot  $G_c(\delta\eta)$  versus  $g_c(\delta\eta)$ , where  $G_c(\delta\eta)$  and  $g_c(\delta\eta)$  are defined by Eqs. (A4) and (A5), respectively. It is seen that the data obtained by numerical simulations with Eq. (1) lies on a straight line with a slope 0.97, which is very close to the theoretically predicted value of 1. Therefore, this analysis provides further support in favor of the loglognormal divergence of  $F_c(\eta)$ . Combining this observation with the result of Ref. [54], we conclude that both disorder in the linear gain/loss coefficient and disorder in the cubic nonlinear gain/loss coefficient lead to the same type of divergence of the  $\eta$ -PDF.

### III. MULTIPLICATIVE-DISSIPATIVE DISORDER IN THE DERIVATIVE CQNLSE

The discussion in Sec. II indicates that the loglognormal divergence of the amplitude PDF is quite general for solitary waves of the CQNLSE model. We now show that similar statistical behavior is exhibited by solitary waves of a second nonlinear wave model. We consider the derivative CQNLSE (DCQNLSE) with weak disorder in the linear gain/loss coefficient

$$i\partial_z \psi + \partial_t^2 \psi + 2|\psi|^2 \psi - \epsilon_q |\psi|^4 \psi + i\epsilon_s \partial_t (|\psi|^2 \psi) = i\epsilon \xi(z) \psi, \quad (15)$$

where  $\xi(z)$  satisfies Eq. (2). In the context of nonlinear optics  $\epsilon_s$  is the self-steepening coefficient and  $i\epsilon_s \partial_t (|\psi|^2 \psi)$  describes the self-steepening effect. In the absence of the perturbation term  $i\epsilon \xi(z) \psi$ , Eq. (15) possesses solitary wave solutions of the form  $\psi_s(t, z) = \Psi_s(x) \exp(i\chi)$ , where

$$\Psi_s(x) = \frac{(2 - \epsilon_s \beta)^{1/2} p}{[(1 - p^2/p_m^2)^{1/2} \cosh(2x) + 1]^{1/2}}, \quad (16)$$

$p = \eta / (1 - \epsilon_s \beta / 2)$ ,  $p_m = [4(\epsilon_q - 3\epsilon_s^2 / 16) / 3]^{-1/2}$ ,  $\chi = \alpha + \beta(t - y) + (\eta^2 - \beta^2)z + g(x)$ , and  $x = \eta(t - y - 2\beta z)$ . In addition, the chirp  $g(x)$  is given by

$$g(x) = - \frac{3}{\sqrt{4}} \epsilon_s p_m \operatorname{arctanh}[B_1 \tanh(x)], \quad (17)$$

where the coefficient  $B_1$  is

$$B_1 = \left[ \frac{1 - (1 - p^2/p_m^2)^{1/2}}{1 + (1 - p^2/p_m^2)^{1/2}} \right]^{1/2}. \quad (18)$$

These solitary wave solutions exist provided that  $\epsilon_q > 3\epsilon_s^2 / 16$  and  $p < p_m$ . Notice that the solitary waves of the DCQNLSE are chirped and are thus fundamentally different from the solitary waves of the CQNLSE.

In the presence of disorder in the linear gain both  $\eta$  and  $\beta$  randomly vary along the propagation. Thus, the dynamics is different from that observed in the CQNLSE case, where only  $\eta$  varies as a result of the disorder. Energy-balance considerations lead to an equation of the form

$$\partial_z \int_{-\infty}^{\infty} dt |\psi|^2 = 2\epsilon\xi(z) \int_{-\infty}^{\infty} dt |\psi|^2. \quad (19)$$

In the first-order adiabatic perturbation procedure we replace  $\psi(t, z)$  with  $\psi_s(t, z)$  in Eq. (19) and obtain

$$\frac{d}{dz} \left[ \operatorname{arctanh} \left( \frac{p}{p_m} \right) \right] = 2\epsilon\xi(z) \operatorname{arctanh} \left( \frac{p}{p_m} \right). \quad (20)$$

Denoting  $\rho_d(z) = \operatorname{arctanh}[p(z)/p_m]$ , we observe that  $\rho_d$  satisfies the stochastic equation

$$d\rho_d/dz = 2\epsilon\xi(z)\rho_d, \quad (21)$$

whose solution in Stratonovich's interpretation is

$$\rho_d(z) = \rho_d(0) \exp[2\epsilon X(z)]. \quad (22)$$

Therefore, the PDF of  $\rho_d(z)$  is lognormal. Changing variables from  $\rho_d(z)$  to  $p(z)$  we obtain that the PDF of  $p$  is given by

$$F_d(p) = \frac{\exp\{-\ln^2[\operatorname{arctanh}(p/p_m)/\rho_d(0)]/(8D\epsilon^2 z)\}}{(8\pi D\epsilon^2 z)^{1/2} p_m (1 - p^2/p_m^2) \operatorname{arctanh}(p/p_m)}, \quad (23)$$

for  $0 < p < p_m$  and  $F_d(p) = 0$  elsewhere. In Appendix B we obtain the same expression for  $F_d(p)$  by using the Fokker-Planck approach. The PDF  $F_d(p)$  has exactly the same form as the PDF of  $\eta$  in systems described by the CQNLSE with disorder in the linear gain/loss coefficient. [Compare Eq. (23) with Eq. (6) in Ref. [54]]. In particular,  $F_d(p)$  exhibits loglognormal divergence in the vicinity of  $p_m$ ,

$$\begin{aligned} F_d(p)|_{p \lesssim p_m} & \simeq \{(8\pi D\epsilon^2 z)^{1/2} \delta p [\ln\{\delta p/(2p_m)\}]\}^{-1} \\ & \times \exp\{-\ln^2\{-\ln[\delta p/(2p_m)]/[2\rho_d(0)]\}/(8D\epsilon^2 z)\}, \end{aligned} \quad (24)$$

where  $\delta p = p_m - p$  and  $0 < \delta p/p_m \ll 1$ . In Appendix C we calculate  $F_d(p)$  by applying Itô's interpretation [58,59] to the stochastic equation (21) satisfied by  $\rho_d$ . We show that in this case as well  $F_d(p)$  exhibits loglognormal divergence near  $p_m$ .

To validate our theoretical predictions we performed Monte Carlo simulations with Eq. (15) with about  $7.8 \times 10^4$  disorder realizations. We used an initial condition in the form of the solitary wave  $\psi_s$  with  $\eta(0)=1$ ,  $\beta(0)=0$ ,  $\gamma(0)=0$ , and  $\alpha(0)=0$  and considered the parameter values  $D=3$ ,  $\epsilon_q=0.7$ ,  $\epsilon_s=0.8$ , and  $\epsilon=0.05$ . For these values,  $p(0)=1$  and  $p_m \simeq 1.13715$ . The simulations were carried out up to a final distance  $z_f=11$ , for which the disorder strength is  $4D\epsilon^2 z_f = 0.33$ . Equation (15) was integrated by employing the split-step method with periodic boundary conditions and with the same numerical scheme as described in Sec. II. The size of the computational domain was taken to be  $-100 \leq t \leq 100$ , and the  $t$ -step and  $z$ -step were taken as  $\Delta t=0.01$  and  $\Delta z=0.001$ , respectively.

The PDF  $F_d(p)$  at  $z=10$  obtained in the numerical simulations is shown in Fig. 4 together with the theoretical prediction. The numerically obtained PDF clearly exhibits divergence in the vicinity of  $p_m$  and the overall agreement between theory and simulation is good. To further check the

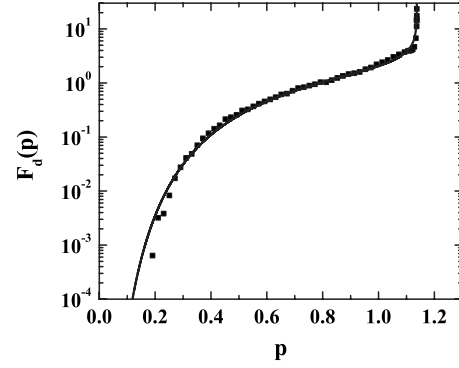


FIG. 4. The probability-density function of  $p$   $F_d(p)$  at  $z=10$  for  $D=3$ ,  $\epsilon_q=0.7$ ,  $\epsilon_s=0.8$ ,  $\epsilon=0.05$ , and  $p(0)=1$ . The squares represent the result of Monte Carlo simulations with Eq. (15), while the solid curve corresponds to the analytic result given by Eq. (23).

behavior of the PDF in the vicinity of  $p_m$  Fig. 5 shows a blowup of the data in the region  $p \lesssim p_m$ . Reasonable agreement between theory and simulations is observed. We attribute the differences between the curves to the difficulties in obtaining an accurate measurement of the group velocity from the numerical data. As a further test for the asymptotic behavior of  $F_d(p)$  near  $p_m$  we employ the procedure for detecting loglognormal divergence that is outlined in Appendix A. Following this procedure we present the graph of  $G_d(\delta p)$  versus  $g_d(\delta p)$  in Fig. 6, where  $G_d(\delta p) = \{-\ln[(8\pi D\epsilon^2 z)^{1/2} \delta p |\ln\{\delta p/(2p_m)\}| F_d(p)]\}^{1/2}$  and  $g_d(\delta p) = (8D\epsilon^2 z)^{-1/2} \ln\{-\ln[\delta p/(2p_m)]\}$ . We observe that the numerically obtained curve lies on a straight line with a slope 0.97, which is very close to the theoretically predicted value of 1. We therefore conclude that the numerically obtained PDF of  $p$  does exhibit loglognormal divergence.

#### IV. DISCUSSION

Here we discuss the underlying reason for the loglognormal divergence of the  $\eta$ -PDF due to disorder in the linear/nonlinear gain/loss. In addition, we briefly discuss a KdV-type of model where the loglognormal divergence of soliton parameters can potentially be observed.

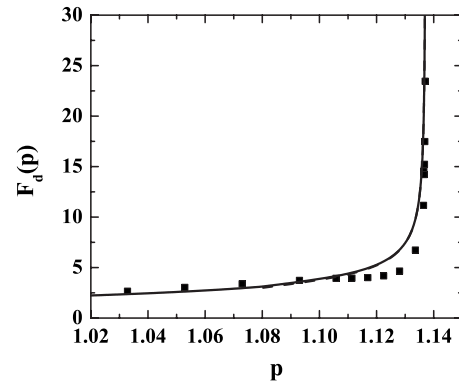


FIG. 5. Blowup of the data shown in Fig. 4 in the vicinity of  $p_m$ . The dashed line corresponds to the asymptotic loglognormal PDF given by Eq. (24).

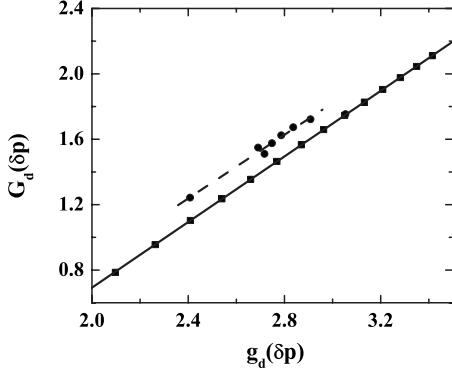


FIG. 6.  $G_d(\delta p)$  vs  $g_d(\delta p)$  for the same parameters considered in Figs. 4 and 5. The squares represent the analytic result obtained with Eq. (23), while the circles stand for the numerical result. The solid and dashed lines are linear fits with slopes 1.00 and 0.97, respectively.

### A. Loglognormal divergence of $F(\eta)$ and superexponential decay of $\delta\eta$ to 0

Consider, for example, the CQNLSE with *deterministic* linear gain,

$$i\partial_z\psi + \partial_t^2\psi + 2|\psi|^2\psi - \varepsilon_q|\psi|^4\psi = i\varepsilon\psi. \quad (25)$$

Using the adiabatic perturbation method we obtain the following equation for the dynamics of  $\eta$ :

$$\frac{d\eta}{dz} = 2\varepsilon\eta_m \left(1 - \frac{\eta^2}{\eta_m^2}\right) \operatorname{arctanh}\left(\frac{\eta}{\eta_m}\right). \quad (26)$$

Even though Eq. (26) can be solved analytically, it is instructive to consider its asymptotic approximation for  $\eta \lesssim \eta_m$ . Denoting  $\delta\eta = \eta_m - \eta$  and expanding both sides of Eq. (26) about  $\eta_m$  while keeping terms up to  $O(\delta\eta)$  we obtain

$$\frac{d\delta\eta}{dz} \approx 2\varepsilon\delta\eta \ln\left(\frac{\delta\eta}{2\eta_m}\right). \quad (27)$$

Integrating Eq. (27) over  $z$  we arrive at

$$\delta\eta(z) \approx 2\eta_m \exp[-\tilde{C}(0)e^{2\varepsilon z}], \quad (28)$$

where  $\tilde{C}(0) = -\ln[\delta\eta(0)/(2\eta_m)] > 0$ . We therefore observe that in this case  $\delta\eta$  decays to 0 superexponentially with increasing  $z$ . It is this superexponential approach of  $\eta$  to  $\eta_m$  that leads to the loglognormal divergence of  $F(\eta)$ . Indeed, for the CQNLSE with disorder in the linear gain/loss coefficient one obtains a similar equation for  $\delta\eta(z)$  with  $z$  replaced by  $X(z)$  on the right-hand side. As a result,  $X(z)$  is related to  $\delta\eta(z)$  via

$$X(z) \approx \ln\left[-\ln\left(\frac{\delta\eta}{2\eta_m}\right) / \tilde{C}(0)\right] / (2\varepsilon), \quad (29)$$

which describes loglognormal divergence of the  $\eta$ -PDF near  $\eta_m$ . A similar result holds for the CQNLSE with disorder in the cubic nonlinear gain/loss coefficient [see Eq. (13)].

### B. Extended Korteweg–de Vries equation

A different type of nonlinear wave equation that possesses flat-top solitary wave solutions is the following extended Korteweg–de Vries (eKdV) equation [3,6,48],

$$\partial_t u + 6u(1-u)\partial_z u + \partial_z^3 u = 0. \quad (30)$$

Note that Eq. (30) is integrable [6,48]. In the context of interfacial waves in two-layer systems and stratified shear flow in the ocean  $u$  represents the vertical displacement,  $z$  is the horizontal coordinate, and  $t$  is time. The solitary wave solutions of Eq. (30) take the form [6]

$$u_s(z,t) = \frac{4\kappa^2}{(1 - \kappa^2/\kappa_m^2)^{1/2} \cosh(2x) + 1}, \quad (31)$$

where  $x = \kappa(z - 4\kappa^2 t)$ ,  $\kappa_m = 1/2$ , and the parameter  $\kappa$  characterizes the soliton amplitude and group velocity.

Since perturbative linear gain and loss terms are quite common in KdV models of wave motion in the ocean [53,62], it is interesting to study the situation where randomness is present. We therefore consider the following perturbed eKdV equation,

$$\partial_t u + 6u(1-u)\partial_z u + \partial_z^3 u = \varepsilon\xi(t)u, \quad (32)$$

where

$$\langle \xi(t) \rangle = 0, \quad \langle \xi(t)\xi(t') \rangle = D\delta(t-t'). \quad (33)$$

Employing mass-balance considerations we obtain that the dynamics of  $\kappa$  is described by an equation similar to Eq. (20). Based on this observation one would expect the  $\kappa$ -PDF to exhibit loglognormal divergence in the vicinity of  $\kappa_m$ . It should be pointed out, however, that for models of the KdV type radiative effects are more significant [62], and as a result, a perturbative calculation that takes these effects into account is required. We therefore defer the full analysis of this case to a future publication.

### V. CONCLUSIONS

We studied the statistics of flat-top solitary wave parameters in the presence of weak dissipative multiplicative disorder. We started by considering propagation of solitary waves of the cubic-quintic nonlinear Schrödinger equation (CQNLSE) in the presence of disorder in the cubic nonlinear gain/loss. We found that the amplitude PDF exhibits loglognormal divergence in the vicinity of the maximum possible amplitude  $\eta_m$ . Since solitary waves with  $\eta$  values near  $\eta_m$  have a typical table-top shape we conclude that the amplitude PDF of flat-top solitary waves is loglognormally divergent. This finding combined with similar findings in Ref. [54] for the case of disorder in the linear gain/loss coefficient indicates that loglognormal divergence is quite ubiquitous for flat-top solitary waves of the CQNLSE in the presence of weak multiplicative dissipative disorder. Furthermore, we showed that this divergence can be explained by the superexponential approach of  $\eta$  to  $\eta_m$  in the corresponding deterministic model with weak linear/nonlinear gain.

Next we considered propagation of solitary waves in the presence of weak disorder in the linear gain/loss in systems

described by the derivative cubic-quintic nonlinear Schrödinger equation (DCQNLSE). The solitary waves of the DCQNLSE are chirped and are thus fundamentally different from the solitary waves of the CQNLSE. As a result of the chirp, in the presence of disorder in the linear gain/loss both the amplitude  $\eta$  and the group velocity  $\beta$  vary randomly along the propagation. We therefore studied the PDF of the parameter  $p$ , where  $p = \eta/(1 - \varepsilon_s \beta/2)$  and  $\varepsilon_s$  is the self-steepening coefficient. We found that  $F_d(p)$  is loglognormally divergent when  $p$  is near its maximum value  $p_m$ , i.e., the  $p$ -PDF of the corresponding flat-top solitary waves exhibits loglognormal divergence. Moreover, we showed that the same divergence is observed in both Stratonovich's interpretation and Itô's interpretation of the linear stochastic perturbation term, thus illustrating another feature of the statistics that appears to be quite general.

### APPENDIX A: THE $G$ VS $g$ METHOD

Here we give the details behind the  $G$  vs  $g$  method that is used to analyze the divergence of the PDFs of  $\eta$  and  $p$  near their maximum possible values. As a specific example we consider the case of the CQNLSE in the presence of disorder in the cubic gain/loss coefficient. In Sec. II we obtained the following approximate analytic expression for  $F_c(\eta)$  near  $\eta_m$ :

$$F_c(\eta)|_{\eta \approx \eta_m} \approx \left\{ (32\pi D \varepsilon^2 z)^{1/2} \eta_m^2 \delta\eta \left| \ln \left[ \frac{e^2 \delta\eta}{2\eta_m} \right] \right| \right\}^{-1} \times \exp \left\{ - \frac{\ln^2[-\ln[(e^2 \delta\eta)/(2\eta_m)]/\tilde{c}]}{32D \varepsilon^2 \eta_m^4 z} \right\}. \quad (\text{A1})$$

The problem we want to address is how to verify that the PDF obtained in the simulations satisfies the loglognormal divergence described by Eq. (A1). In particular, we need a method that will allow us to ignore the coefficient  $\tilde{c}$ , which cannot be found from the numerical data. Furthermore, we would like to find a mapping that "stretches" the small  $\eta$  neighborhood of  $\eta_m$  into a wider interval. To address these issues we rewrite Eq. (A1) in the form

$$\ln \left[ (32\pi D \varepsilon^2 z)^{1/2} \eta_m^2 \delta\eta \left| \ln \left( \frac{e^2 \delta\eta}{2\eta_m} \right) \right| F_c(\eta) \right] \approx - \frac{\ln^2\{-\ln[(e^2 \delta\eta)/(2\eta_m)]/\tilde{c}\}}{32D \varepsilon^2 \eta_m^4 z}, \quad (\text{A2})$$

where it is understood that  $F_c(\eta)$  is calculated near  $\eta_m$ . Multiplying by  $-1$  and taking the square root we arrive at

$$\left\{ - \ln \left[ (32\pi D \varepsilon^2 z)^{1/2} \eta_m^2 \delta\eta \left| \ln \left( \frac{e^2 \delta\eta}{2\eta_m} \right) \right| F_c(\eta) \right] \right\}^{1/2} \approx \frac{\ln\{-\ln[(e^2 \delta\eta)/(2\eta_m)]\} - \ln(\tilde{c})}{(32D \varepsilon^2 \eta_m^4 z)^{1/2}}. \quad (\text{A3})$$

We now define  $G_c(\delta\eta)$  and  $g_c(\delta\eta)$  as

$$G_c(\delta\eta) = \left\{ - \ln \left[ (32\pi D \varepsilon^2 z)^{1/2} \eta_m^2 \delta\eta \left| \ln \left( \frac{e^2 \delta\eta}{2\eta_m} \right) \right| F_c(\eta) \right] \right\}^{1/2}, \quad (\text{A4})$$

and

$$g_c(\delta\eta) = \frac{\ln\{-\ln[(e^2 \delta\eta)/(2\eta_m)]\}}{(32D \varepsilon^2 \eta_m^4 z)^{1/2}}. \quad (\text{A5})$$

Using these definitions we observe that when the statistics of  $\eta$  is described by Eq. (A1) the graph of  $G$  vs  $g$  is a straight line with a slope 1, independent of the value of  $\tilde{c}$ . A similar conclusion (with slightly different expressions for  $G$  and  $g$ ) holds in the case of DCQNLSE with disorder in the linear gain/loss coefficient.

### APPENDIX B: FOKKER-PLANCK APPROACH FOR CALCULATION OF THE PDFS

In this appendix we demonstrate that the expressions for the PDF of the solitary wave parameters that were obtained in Secs. II and III by solving the Langevin equation can also be obtained within the framework of the Fokker-Planck approach. As a specific example we consider the DCQNLSE with disorder in the linear gain/loss coefficient and work with Stratonovich's interpretation of Eq. (21). The corresponding Fokker-Planck equation for the PDF of  $\rho_d$ ,  $H(\rho_d, z)$ , is [58,59]

$$\partial_z H = 2D \varepsilon^2 \partial_{\rho_d} [\rho_d \partial_{\rho_d} (\rho_d H)]. \quad (\text{B1})$$

Changing variable to  $w = \ln \rho_d$  we arrive at

$$\partial_z \tilde{H} = 2D \varepsilon^2 \partial_w^2 \tilde{H}, \quad (\text{B2})$$

where

$$\tilde{H}(w, z) = \rho_d(w) H(\rho_d(w), z). \quad (\text{B3})$$

The solution of Eq. (B2) with the initial condition  $\tilde{H}(w, z) = \delta[w - w(0)]$  is

$$\tilde{H}(w, z) = \frac{\exp\{-[w - w(0)]^2/(8D \varepsilon^2 z)\}}{(8\pi D \varepsilon^2 z)^{1/2}}. \quad (\text{B4})$$

Changing variable from  $\rho_d$  to  $p$  while using Eqs. (B3) and (B4) we obtain

$$F_d(p) = \frac{\exp\{-\ln^2[\text{arctanh}(p/p_m)]/\rho_d(0)\}/(8D \varepsilon^2 z)\}}{(8\pi D \varepsilon^2 z)^{1/2} p_m (1 - p^2/p_m^2) \text{arctanh}(p/p_m)} \quad (\text{B5})$$

for  $0 < p < p_m$  and  $F_d(p) = 0$  elsewhere. Equation (B5) is the same as Eq. (23) in Sec. III. A similar calculation based on the Fokker-Planck approach leads to Eq. (10) for the  $\eta$ -PDF for the CQNLSE with disorder in the cubic gain/loss coefficient.

### APPENDIX C: CALCULATION OF $F_d(p)$ IN ITÔ'S INTERPRETATION

Consider the DCQNLSE with disorder in the linear gain/loss coefficient. We now obtain the PDF of  $p$  by employing

Itô's interpretation to Eq. (20), and show that this PDF exhibits loglognormal divergence in the vicinity of  $p_m$ . The solution of the equivalent Eq. (21) in Itô's interpretation is [58]

$$\rho_d(z) = \rho_d(0) \exp[2\epsilon X(z) - 2\epsilon^2 z], \quad (\text{C1})$$

where the PDF of  $X(z)$  is given by Eq. (9). Therefore,  $X(z)$  is related to  $p(z)$  via

$$X(z) = \frac{1}{\epsilon} \left\{ \ln \left\{ \operatorname{arctanh} \left[ \frac{p(z)}{p_m} \right] / \rho_d(0) \right\} + 2\epsilon^2 z \right\}. \quad (\text{C2})$$

Changing variables from  $X(z)$  to  $p(z)$  we obtain the PDF of  $p$  in Itô's interpretation:

$$F_d^{(l)}(p) = [(8\pi D\epsilon^2 z)^{1/2} p_m (1 - p^2/p_m^2) \operatorname{arctanh}(p/p_m)]^{-1} \\ \times \exp\{-[\ln\{\operatorname{arctanh}(p/p_m)/\rho_d(0)\} \\ + 2\epsilon^2 z]^2 / (8D\epsilon^2 z)\}. \quad (\text{C3})$$

In the vicinity of  $p_m$  Eq. (C2) can be approximated by

$$X(z) \approx \frac{1}{\epsilon} \left\{ \ln \left\{ -\frac{1}{2\rho_d(0)} \ln \left[ \frac{\delta p(z)}{2p_m} \right] \right\} + 2\epsilon^2 z \right\}. \quad (\text{C4})$$

Consequently,  $F_d^{(l)}(p)$  is given by

$$F_d^{(l)}(p)|_{p \approx p_m} \approx \{(8\pi D\epsilon^2 z)^{1/2} \delta p |\ln[\delta p/(2p_m)]|\}^{-1} \\ \times \exp\{-[\ln\{-\ln[\delta p/(2p_m)]/[2\rho_d(0)]\} \\ + 2\epsilon^2 z]^2 / (8D\epsilon^2 z)\}, \quad (\text{C5})$$

which exhibits loglognormal divergence as  $p$  approaches  $p_m$ . We therefore conclude that for the DCQNLSE with disorder in the linear gain coefficient both Stratonovich's interpretation and Itô's interpretation lead to loglognormal divergence of the PDF of  $p$ .

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